The Diffusion Equation Does Not Imply Instantaneous Action at a Distance

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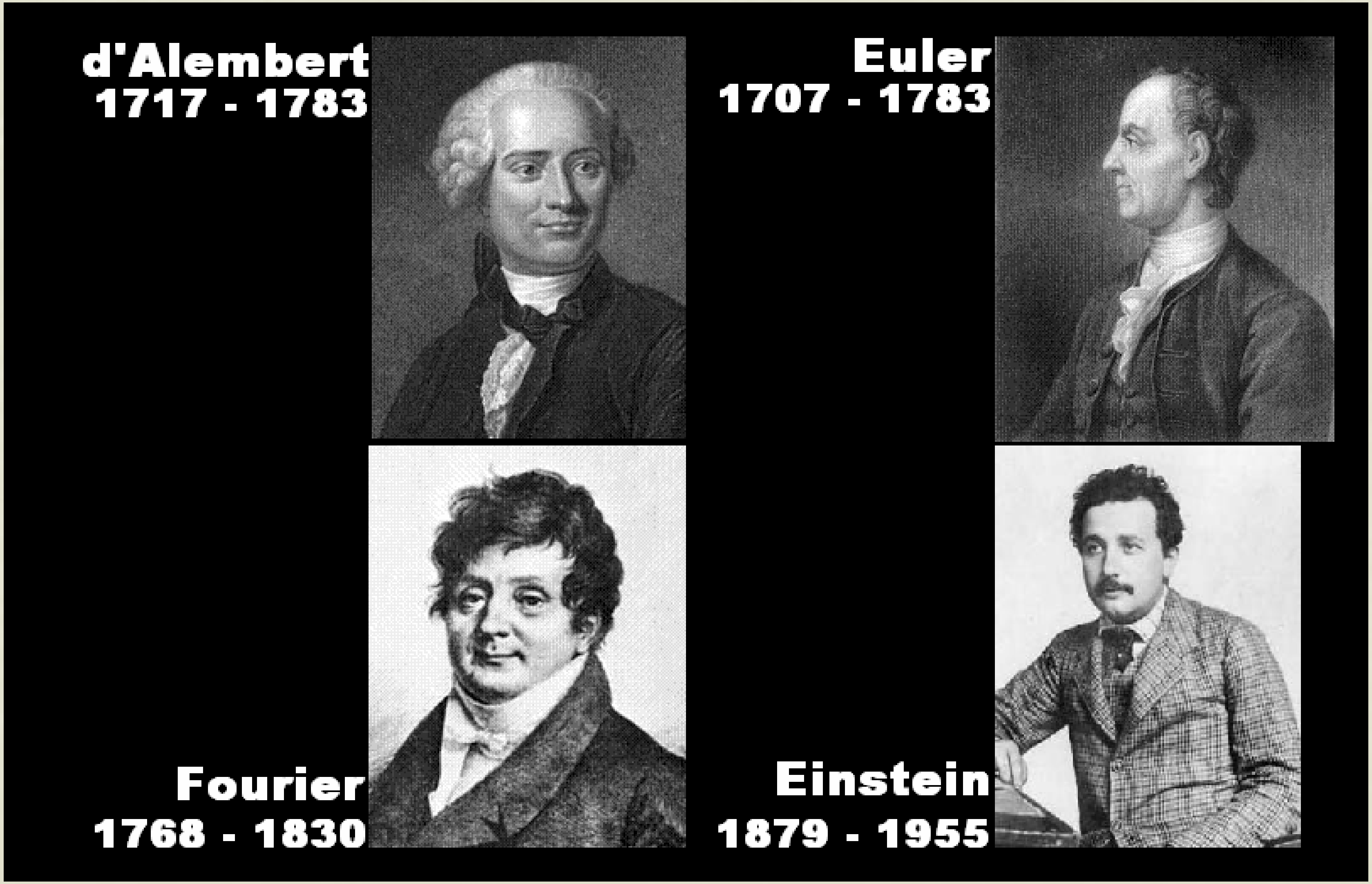
The Men…


d'Alembert
1717 - 1783
Euler
1707 - 1783
Fourier
1768 - 1830
Einstein
1879 - 1955

Outline

- About the Diffusion Equation
- About Instantaneous Action at a Distance (IAD)

- Fourier's (Classical) Theory of Diffusion
  - Why IAD arises in the Fourier theory

- Einstein's (Stochastic) Analysis of Diffusion
  - How to introduce IAD in one of Einstein's own theories!!

- A Couple of Novel Questions
  - And the discriminant to tell if IAD is present in a theory or not

- Concluding Remarks

The Diffusion Equation

$$\nabla^2 \theta = \frac{1}{\kappa} \frac{\partial \theta}{\partial t}$$

- One of the most fundamental equations
- Assumes a flux-to-gradient functional relationship
  - e.g. Fourier's law of heat conduction
- Assumes some kind of conservation principle
  - e.g. conservation of energy
- Applicable in a wide range of situations
  - Diffusion of Mass
  - Conduction of Heat
  - Spread of Vorticity in a Viscous Fluid
  - Propagation of Electric Field Vector in a Long Conductor
  - Formally, Schrödinger's Equation of Quantum Mechanics

Instantaneous Action at a Distance (IAD)

- Toss the Sun suddenly off its course. When will the Earth get to feel the change? Instantaneously?

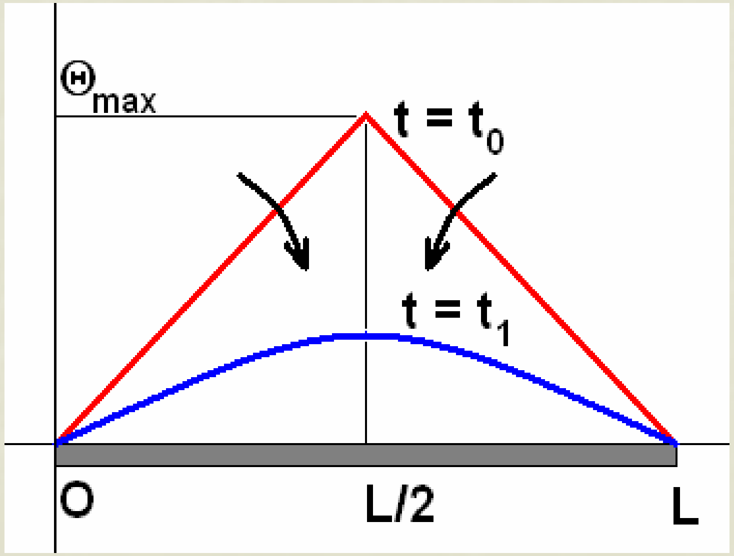
- Einstein called the idea of IAD "spooky"
  - He took it that no material influence could move faster than light
- Yet, something like quantum entanglement does exist

- And then, IAD is relevant in classical theories too
  - Think: Does a straight lever-arm remain straight?

- IAD is said to "reveal a flaw in our diffusion equation"
  - But to say so is an inversion—i.e., a logical reversal
  - The flaw is in the Fourier technique, not in the diffusion equation

IAD, in the Analytical Theory of Heat (!)

- These days, IAD in diffusion is not even mentioned in textbooks
  - The rare exceptions are: Greenberg (1998), Shercliff (1977), Morse and Feschback (1953)
  - But not in thousands of other books: Feynman, Kreyszig, Wylie & Barrette, Mathews & Walker, Churchill & Brown, Sneddon, Holman, Lienhards, etc.
- If at all mentioned, the examples given are for infinite domains
- Yet, IAD is possible in finite domains as well!
- E.g.: The sharp tip in the temperature distribution is lost in an infinitesimally small time



Fourier's Theory of Heat (1807 AD)

- d'Alembert: Invented Separation of Variables (1749)
- Euler: Gave formulae for the "Fourier coefficients" (1777)
- Fourier: Expressed the sought solution in the form of a series of separated variables (1807)

$$\theta(x,t) = \sum_{j=0}^{\infty} \exp\left(-\alpha^2 \kappa\, t\right)\left[C_{xj} \cos(\alpha x) + S_{xj} \sin(\alpha x)\right]$$

- Overall Idea: Express the initial temperature distribution itself as an infinite series of <u>spatial</u> harmonics
- For each wavenumber in the series:
  - The time-dependent part has an exponential decay
  - The space-dependent part has cosines and sines whose coefficients are found from the initial value, using Euler's formulae

The Origin of IAD in Fourier's Theory

- So, first, use the Fourier series/integral, to express the initial value distribution

$$\theta(x,t) = \sum_{j=0}^{\infty} \exp\left(-\alpha^2 \kappa\, t\right)\left[C_{xj} \cos(\alpha x) + S_{xj} \sin(\alpha x)\right]$$

Then the question: How about its evolution in time?

- Each spatial harmonic suffers the exponential decay independent of others, and with a different half-life
- The decay occurs at all points of domain, <u>simultaneously</u>
- …In a way, this is just like the ever-straight lever arm!
  - Which means, IAD!!
- IAD is built into the technique of separation of variables
- Therefore, in Fourier's theory

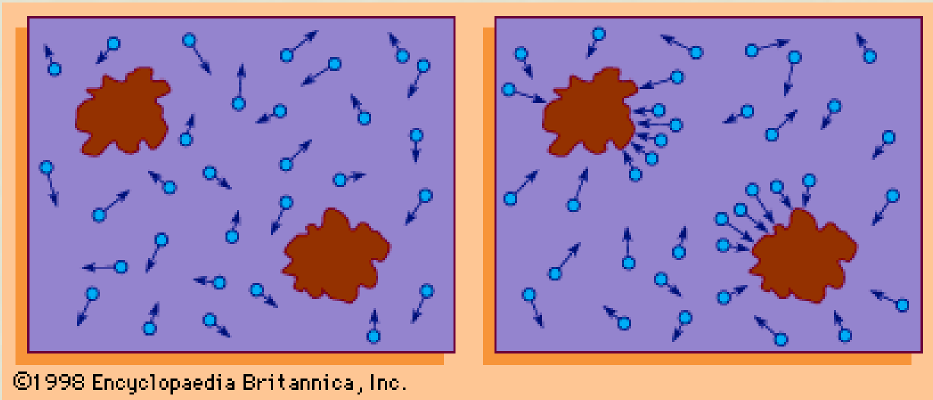
Points to Note about the Classical Theory

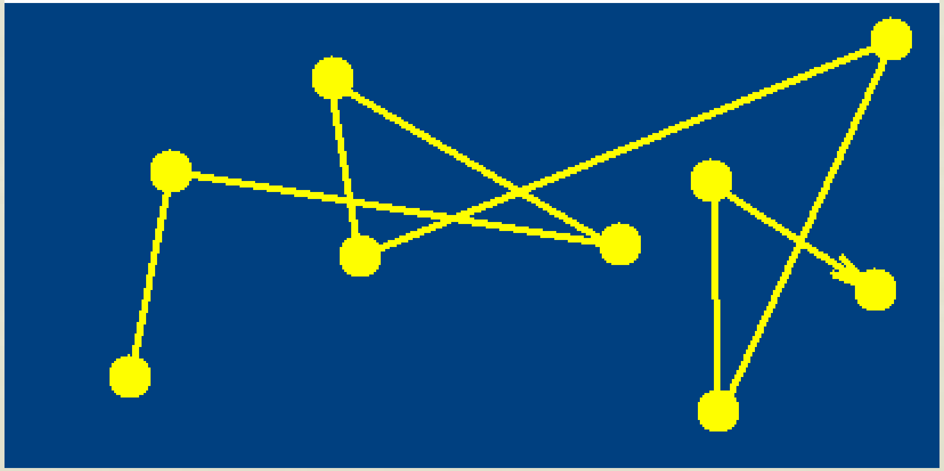
- The Fourier series does not give a general solution to any PDE—whether the equation is diffusion, wave…
- The Fourier theory does not solve the diffusion equation
- The Fourier theory only re-models the diffusion problem in an analytically more tractable form
  - But this compromise is no longer unnecessary—Computers

- Absurdities resulting from IAD in diffusion: Examples
  - Thermit welding of rails in Delhi will heat up the railroad in Visakhapatnam right at the same time—or, vice versa
  - If you touch just a corner of a blotting paper with ink, the entire paper will turn blue in no time!
  - As soon as you open a bottle of scent in India, some of it will appear in the USA—precisely at the same time!
  - And, the abundance of conceptual confusions in quantum mechanics! (Schrödinger's equation is a diffusion equation)

A Second Kind of Theory—Stochastic Theory

- Einstein (1905)
- If
  - the instantaneous motions
  - of the same particle
  - observed at two successive instances
  - are mutually independent
  - and random
- Then
  - the relative displacement variable obeys the diffusion equation
- The "particle" can be a molecule itself





The Diffusion Equation in Einstein's Theory

$$\frac{\partial^2 N}{\partial \xi^2} = \frac{1}{D}\frac{\partial N}{\partial t} \qquad \text{where} \qquad D = \frac{\langle \xi^2 \rangle}{2\tau}$$

$$\text{and} \qquad N(x, t+\tau)dx = dx \cdot \int_{-\infty}^{+\infty} N(x+\xi, t) f(\xi) d\xi$$

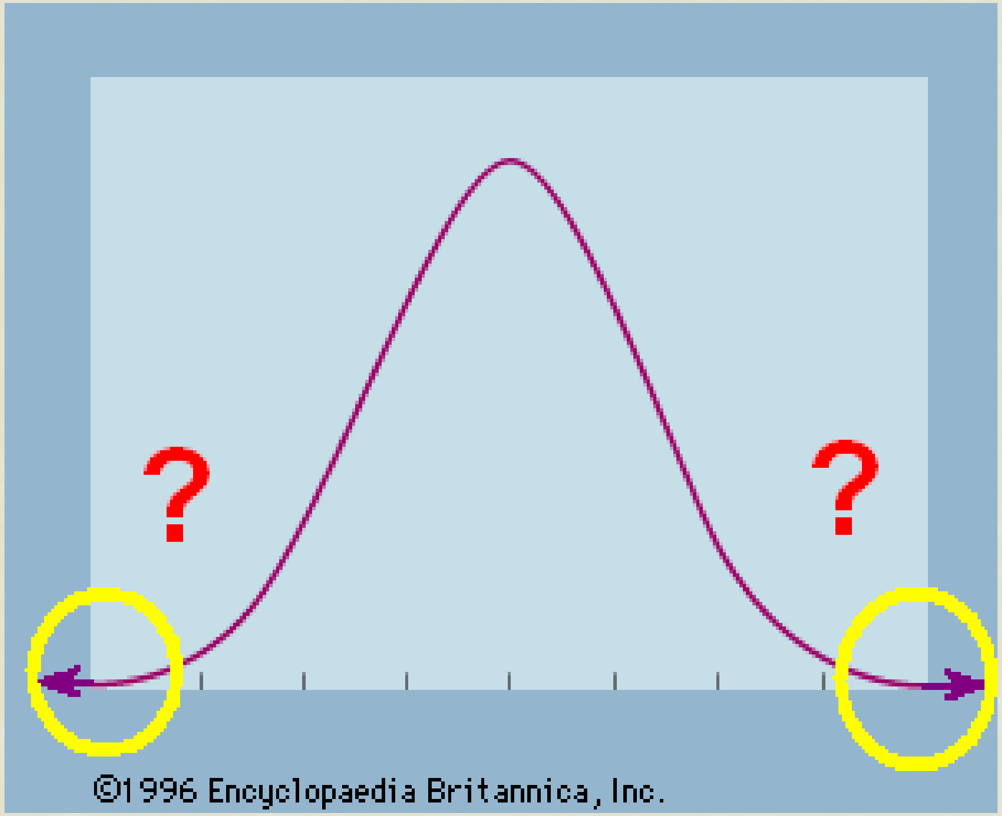
- N is the instantaneous density of particles at a point
- ξ stands for relative displacements (from x = xo)
- ξ is a continuous random variable. The associated probability density function (PDF) is the "f(ξ)" term
- Important: The diffusion law is insensitive to the particular form that the PDF f(ξ) assumes

The Importance of the Support of f(ξ)

- Einstein anticipated f(ξ) to "differ from zero only for very small values of" ξ
  - This amounts to assuming a finite support for f(ξ)

- But what if the support of f(ξ) is infinite?

- Physically this means: the suspended particle can travel arbitrarily large distances in arbitrarily small time



How to Detect IAD in a Theory

- What determines if a given theory implies IAD or not?

- The answer does <u>not</u> depend on whether:
  - the theory is classical or stochastic
  - the diffusing substance is continuous or discrete
  - the PDF $f(\xi)$ is continuous or discrete
  - the variable ξ is random or deterministic
  - the domain is finite or infinite in extent
  - the <u>initial value</u> is spread over finite or infinite portion

- What does matter is: The <u>compactness of the support</u> of the PDF $f(\xi)$,
  - Or, somewhat equivalently, that of the Dirac pulse
  - Thus, what matters is the "width" of $f(\xi)$, inclusive of its tails
  - The criterion can serve as a discriminant <u>← New Development</u>

A Couple of Novel Questions

- If IAD is implied by the Fourier Theory, how come it doesn't come up in Einstein's theory? Or does it?
  - It seems Einstein took the support of $f(\xi)$ to be finite
  - But if you make the support infinite (say, via the Gaussian distribution), you will have IAD in Einstein's own theory too!
- Would it be possible to remove IAD from Fourier's theory?
  - In principle, yes!
  - But this seems like a rather contrived procedure (as of today)
- What Else?
  - The wavelets theory seems <u>not</u> very promising for the purpose ("Scale" is not a proper generalization of "Fourier Frequency")
  - A wavelets-based description may still carry IAD

What We Discussed Here…

- We discussed IAD in a primarily classical context—neither relativistic nor quantum mechanical
- We gave an example of IAD in a finite domain

- We pointed out the following
  - The compactness of support of f(ξ) as the discriminant ← New!
  - IAD can be introduced in one of Einstein's own theories ← New!
  - IAD can be removed from the Fourier theory ← New!
  - Clarification of the proper hierarchy (or conceptual order): IAD → Separation of Variables → Fourier's Theory → "Solutions" of the Diffusion Equation

- It seems nobody made these observations anytime before, i.e., right since Fourier's time (1807)

So, What? … What's the Point?

- Fourier's theory is very important—practically speaking
  - "Fourier—the Father of Modern Engineering," Eugene F. Audiutori, Mechanical Engineering, ASME (August 2005)
  - The theory is useful in <u>every</u> branch of engineering
  - It has become one of <u>the most widely used</u> mathematical tools in the history of physics
  - Yet, all the confusions we saw exist right at its core (or at its bases)
  - So, it has also become the most widely <u>abused</u> mathematical tool!

- So, the point is:
  Do you want to remain slightly wrong all the time?
  - i.e., wrong in <u>conceptual</u> terms—not just as a numerical approximation that can in principle be refined indefinitely

- This work serves to lay the groundwork for 'FAQ'…
- …and also, for discussing quantum entanglement
  - Aside: There is no need to have IAD in QM either!

Thank You!

- Credits
  - The photographs in slide # 2 were taken from the Web pages on history of mathematics maintained by the School of Mathematics and Statistics, University of St. Andrews, Scotland. URL: http://www-groups.dcs.st-and.ac.uk/~history/index.html
  - The top diagram in slide # 10 and the original diagram in slide # 12 both came from Encyclopaedia Britannica, year 2000 CD edition

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