A New Approach for Modeling the Ideal Fluid Flow on Computer*

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A new numerical approach is being developed *ab initio* to address the Helmholtzian class of engineering field problems. The new approach does not involve simultaneous equations as in FEM or FDM. Instead, it theoretically begins by viewing Huygens' Principle in the context of geometric probability. This yields a modeling approach similar to but more general and practical than prior studies involving random walk. A theorem fundamental to the new approach is stated and conceptually explained. Trials for the new approach are conducted on personal computer for certain planar flows, and their results compared to both closed-form analytic solution and FEM study. These early results are highly encouraging. Some of the main features of the new approach relevant to its application in engineering practice are pointed out.

I. THE IDEAL FLUID FLOW

Consider an infinitesimal fluid element at a point in space in a flowing fluid. According to the principle of mass conservation, the rate of change of mass inside the infinitesimal cube is balanced by the net rate of mass efflux out of the control surfaces. Neglecting the higher order nonlinear terms in the Taylor series expansion for velocity variations across the elemental cube, the following expression of mass conservation, known as the continuity equation, is obtained [1]:

$$\overrightarrow{\nabla} \cdot \rho \vec{V} = -\left(\frac{\partial \rho}{\partial t} + \dot{m}\right) \tag{1}$$

where $\overrightarrow{\nabla}$ is the vector operator del; the symbol $\cdot$ denotes the dot product of vectors; ρ is the density and $\overrightarrow{V}$ is the velocity of the fluid; $t$denotes time; and $\dot{m}$ is the time rate of mass introduction at any singularity contained within the element.

For a flow that is all of the following: irrotational, incompressible, and steady-state, the continuity equation reduces to:

$$\nabla^2 \phi = -M \tag{2}$$

where ∇^2 is the Laplacian operator; ϕ is a scalar function called the velocity potential; and M is the steady strength of singularity. The essential idea here is that the velocity field can be obtained as the gradient of a scalar velocity potential ϕ, because the curl of the velocity field is zero for an irrotational, steady-state, (i.e., by implication, inviscid) and incompressible flow. Equation (2) describes the so-called ideal fluid flow. It is Helmholtzian in form [5] and so, a homogeneous, linear, second order partial differential equation. Being linear, its individual solutions can be superimposed [2]. Several standard texts show how the elementary plane flows can be combined to yield solutions for the situations of practical interest; e.g., refer [3].

II. A CONCEPTUAL DEVELOPMENT OF THE THEOREM FUNDAMENTAL TO THE NEW APPROACH

Consider the problem of wave propagation in a continuum. This problem shares the Helmholtzian form with the ideal fluid flow [5]. According to Huygens' Principle, the spatial distribution of wave-fronts is obtained by imagining the particles at any wave-front as sources of new circular waves [7]. The principles of geometric probability [8] can be applied to this situation.

Consider a source in an infinite space radiating waves outwards. Each particle on an arbitrarily chosen wave-front radiates small, circular, Huygens waves. Now, if the original wave-front is sampled statistically at random to yield a point, and then, the Huygens wave from this point also further sampled, then a displacement vector connecting the two sampled points is obtained. Now, imagine that sampling begins at source and proceeds through all the successive secondary waves. Due to randomness of sampling, if all these displacement vectors are joined in sequence, the overall pattern is a random walk "motion." The complete wave field is obtained by superimposing all such random walks. The continuum solution is approached in the limit when (i) an infinite number of particles originate at the source; and (ii) the displacement vector for each pair of sampling is infinitesimally small.

By suitable mathematical manipulation involving appropriate field variable(s) and the time variable, the technique may in principle be applied to any Helmholtzian field problem, regardless of whether it is elliptic, parabolic or hyperbolic in formal classification. The idea here is that the above procedure says something essential about the Laplacian operator itself. This premise is stated as a theorem fundamental to the new approach:

The superposition of an infinite number of particles undergoing infinitesimally small displacements in random-walk is a solution to any Helmholtzian field problem.

For on-computer implementation, both factors in the above theorem are finite: (i) the number of particles (N), and (ii) the size of the displacement vector for the sampled waves (R). Therefore, computer trials yield probabilistic (i.e. "graded") estimates of—or approximate solutions to—the governing differential equations; the equations themselves having been cast in the continuum form. The solution accuracy of a trial (i.e. the quality of

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its conformance as a solution to the governing equation) increases with increasing N and decreasing R.

The on-computer implementation in this study follows from the above basic principles. While modeling engineering problems, there arise several further considerations such as finite objects, arbitrary geometry, specification of initial values and boundary conditions (Dirichlet, Neumann, mixed), etc. This paper is mainly concerned with announcing the new approach and presenting the early results; a detailed discussion of these subsequent considerations is out of its scope.

The approach outlined here was developed completely independent of (and indeed, without any prior knowledge of) the previous work done elsewhere such as Witten's classic paper of 1981 on diffusion-limited aggregation (DLA), reprinted and reviewed in [9]. Other prior work, e.g., that referenced in [10], does acknowledge that in theory a random walk of infinitesimally small steps conforms to the Poisson equation. (It also implies infinite energy—a questionable conclusion. Yet, due weight is not given to the condition regarding an infinite number of particles.)

In contrast, the present work begins with Huygens' Principle, and then takes the view that the resultant approach represents a general method of engineering field analysis; both these are new developments. The published studies concerning on-computer trials of random walk for the past two decades have exclusively focused on the chaotic or fractal aspects of DLA, fluid motion, etc., where the appropriate fractal dimension, d, is less than 2; not on how to solve the Helmholtzian field problems in engineering where d equals 2. Further, prior computer trials are concerned only with the infinite planar domain; not with the engineering considerations of finite 3-D objects of arbitrary geometry, and arbitrary initial values and boundary conditions. Thus, there is little surprise that comparison to FEM has not been undertaken before. Overall, from an engineering standpoint, prior work is little more than a theoretical or mathematical curiosity.

III. TRIALS ON COMPUTER FOR COMPARISON OF THE NEW APPROACH TO ANALYTIC SOLUTION AND FEM

A. Introduction

The results of the new approach are compared to the closed form analytic solution and FE analysis using two separate case-studies. In all cases, modeling is done and analysis conducted on the same, moderately powerful, personal computer (Intel Pentium III at 933 MHz clock speed and 512 MB RAM). The development environment is Microsoft Visual C++ 6.0 on Microsoft Windows 2000 Professional operating system. The current software implementation is not optimized in any sense such as float-versus-integer representation; run-time reordering of data structures; caching pre-computed data; multi-threading; etc.

B. Comparison to the Closed Form Analytic Solution: Infinite Planar Flow

Figure 1 shows the results of modeling via the new approach for the case of infinite planar flow into a sink at the center, under the ideal fluid flow assumptions. The closed-form analytic solution of Equation (2) for this particular case of the ideal fluid flow is given [3] as:

$$\phi = \frac{q}{2\pi} \ln(r) \tag{3}$$

where ϕ is the velocity potential; q is the strength of singularity (i.e. the volume flow rate per unit depth); and r is the radial distance outwards from the sink at the center.

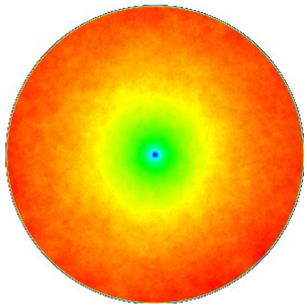

FIG. 1: The Infinite Planar Flow (with a Sink at the Center) Modeled Using the New Approach

Figure 1 is a reduced-size reproduction of the analysis results using the new approach. It depicts the velocity potential field. The radius of the circle is 150 mm. The R parameter is constant at 1 mm. The picture uses 32768 different shades of color from red for high to blue for low magnitudes. The analysis was carried out for 30 minutes. The data become increasingly more accurate with longer execution time. The refinement speed depends on the size of the domain and the R parameter. The R parameter, in turn, is selected as per the desired accuracy.

Table I gives the velocity potential at uniformly spaced 11 points along the horizontal radial vector starting from the sink and growing out towards the right edge. A velocity potential (ϕ) value of 1000 mm is assumed to exist at the outer periphery, i.e., at the radial distance of 150 mm. A flow rate (q) of 2π mm-square/s is assumed for simplicity of quantitative comparison. The last row in Table I is taken at an irregular interval to ignore the edge effect near the periphery. The edge effect arises due to finitization in on-computer trials in the new approach.

C. Comparison to FEM Solution: Finite Planar Cavity

The problem of modeling the 2-D flow past a square hole inside a finite square cavity is chosen because studies of such geometries is a standard practice in the FE modeling of fluid flow [4, 5]. This relatively simple problem still involves sufficient complexity that the closed-form analytic solutions are not available. The domain is in the form of a square of 60 cm sides with a central square hole of 20 cm sides; see Figure 2 and Figure 3. For all cases, the left side of the domain is held at a constant velocity potential of 1000 mm and the right side at 0

Sr. Num.	Distance from the sink, mm	Velocity Potential, mm		Percentage Difference of Modeled from Predicted
		As Modeled	Predicted	
1	0	70.41	0	N.A
2	15	551.12	553.71	-0.47
3	30	669.29	685.80	-2.41
4	45	745.88	764.61	-2.45
5	60	801.20	820.98	-2.41
6	75	835.10	864.88	-3.44
7	90	886.69	900.86	-1.57
8	105	910.65	931.33	-2.22
9	120	928.47	957.76	-3.06
10	135	928.93	981.10	-5.32
11	148	943.36	997.99	-5.47

TABLE I: Comparison of the New Approach with the Closed Form Analytic Solution

mm. The top and bottom sides of the domain, and all the sides of the square obstacle, are impermeable.

The triangular element FEM mesh is generated using the "EasyMesh" program by Niceno [6]. The element side is 2 cm along the outer boundaries and 1 cm along the internal boundaries. This results in a non-uniform mesh consisting of 1693 nodes and 3186 triangular elements.

The C++ source code for FE analysis follows closely along the lines suggested by [4]. The FE modeling here uses the elimination approach for incorporating the nodal boundary conditions in the matrix equation [5]. The matrix solution for the unknown nodal potentials is obtained using the standard Gaussian elimination technique with full pivoting and back-substitution.

Overall, in this study, FE analysis mainly serves as a benchmark for examining the validity of the new approach. The run-time comparisons presented here are to be taken only in an indicative sense.

Figure 2 shows the results of the FE analysis along with the mesh used in it. Figure 3 shows the results following the new approach. The R parameter is randomly varied between 0.2 and 2 cm. The subsequent smoothening is carried out with a block size of 2 cm. Both Figure 2 and Figure 3 depict the velocity potential field using 32768 shades of color ranging from red to blue in the decreasing order of magnitude.

The FEM analysis took about 90 seconds to complete; hence, modeling with the new approach was also run for 90 seconds. However, in the case of FEM, the mesh generation itself took an additional time of about 30 seconds. The new approach does not require mesh generation. When analysis with the new approach is run for longer time, the potential field is better refined. However, the comparative data reported in this paper only refer to the trials having identical times for the analysis stage itself.

Table II gives the nodal values of the velocity potential at those FEM nodes that fall closest to a horizontal line at y = 18; i.e. the line passing across the entire domain a little below the sharp-angled square obstacle. This line passes through the regions of greatest changes in the field variable. For brevity in reporting, only some of such nodes (selected at the nearest 5 cm intervals to ensure the absence of a systematic bias) are shown in Table II.

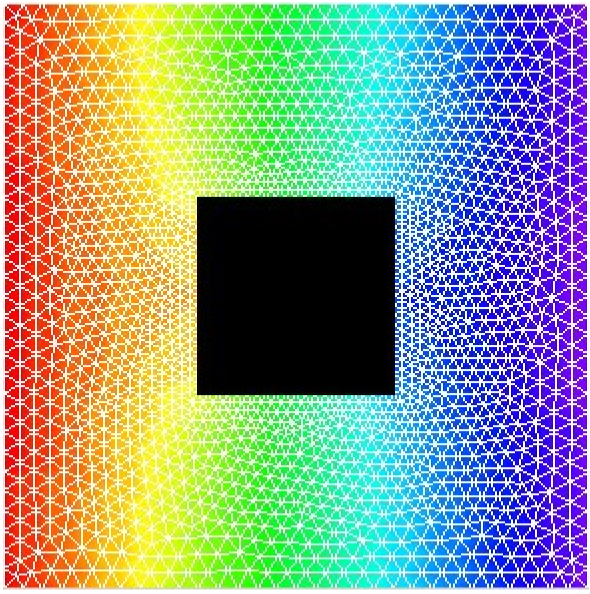

FIG. 2: Velocity Potential Field using Finite Element Analysis

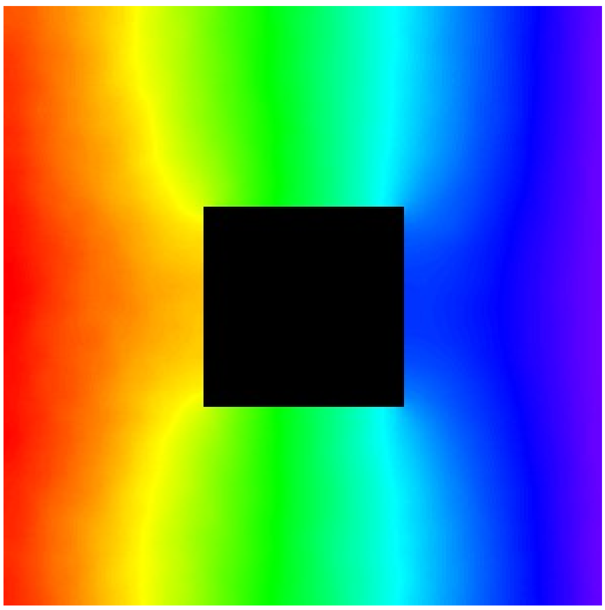

FIG. 3: Velocity Potential Field using the New Approach

FEM Node	Coordinates, cm X	Y	Velocity Potential, mm FEM	New Approach Original	Smooth	Percentage Difference from FEM Original	Smooth
1	00.00	18.00	1000.00	947.71	977.49	-5.23	-2.25
2	6.15	18.20	922.84	871.73	906.38	-5.54	-1.78
3	10.19	17.98	871.12	825.16	853.69	-5.28	-2.00
4	15.13	17.83	804.52	739.92	794.63	-8.03	-1.23
5	19.93	18.19	725.45	681.92	720.88	-6.00	-0.63
6	24.95	18.23	608.69	553.92	604.06	-9.00	-0.76
7	30.06	18.14	498.74	504.90	500.07	+1.24	+0.27
8	35.93	18.23	371.59	358.66	381.39	-3.48	+2.64
9	40.11	18.14	274.16	263.89	285.15	-3.75	+4.01
10	43.48	18.24	212.83	204.25	218.64	-4.03	+2.73
11	51.70	18.05	104.58	105.94	115.57	+1.29	+10.50
12	56.61	18.06	42.71	54.19	54.46	+26.90	+27.51
13	60.00	18.00	0.00	10.89	16.68	NA	NA

TABLE II: Comparison of the New Approach with the FEM Results

IV. A DISCUSSION OF THE RESULTS

A. Comparison with the Closed-Form Solution

Table I shows excellent agreement between the theoretically predicted field values and their approximate counterparts as computed using the new approach. The last column of Table Igives the error estimates from the analytic solutions.

The error magnitudes are smaller than $\pm 5\%$ over the entire domain except near the right edge. Thus, the errors are within the acceptable range for a numerical method. The edge effect arises due to finitization in the new approach.

B. Comparison with the FEM Results

Referring to Table II, the agreement between the results of the new approach and those by FEM is again very good. Even the un-smoothened values of the new approach do not differ from the FEM nodal values by more than$\pm 10\%$ over most of the domain. The smoothened values are in even closer agreement, within $\pm 5\%$ for the most part. The percentage differences of the new approach over the FEM results show a pronounced departure near the sink. However, the differences are not so significant in absolute terms, i.e. against the range of 0 to 1000. In future, it should be possible to reduce these localized errors using a better implementation technique. Further, a better smoothening operator may render the smoothened values even closer to FEM. It should also be noted that FEM results themselves are approximate due to the finitude of the elements.

Indeed, the new approach agrees better with the closed-form analytical predictions (Table I) than it does with the FEM results despite using a non-uniform dense mesh (Table II).

V. SOME OF THE SALIENT FEATURES OF THE NEW APPROACH

The following are some of the salient features of the new approach; the list is only indicative.

1. Simultaneous equations are not used. Matrix manipulations are not involved.
2. The steps of mesh generation and/or its subsequent refinements are not involved.
3. The analyst can be kept periodically updated about the solution refinements during analysis.
4. Handling singularities requires no special treatment.
5. It is inherently parallelizable, requiring no modification whatsoever to the main algorithm.
6. Yet, a single, moderately powerful, personal computer can also easily handle the new approach without unduly long analysis time.

VI. CONCLUSIONS

1. A new numerical approach based on the application of the geometric probability techniques to Huygens' Principle is independently formulated. This approach is taken to be general enough to allow engineering analyses of any Helmholtzian field problem.
2. A theorem fundamental to the new approach is stated and its conceptual roots outlined.
3. The new approach is applied to the problem of the ideal fluid flow. Trials are conducted on a moderately powerful PC, and the results of the new approach compared in two cases:
   (a) The closed-form analytic solution for an infinite planar flow having a singularity
   (b) FE analysis for the finite planar flow past a square obstacle in a square cavity.
4. The on-computer trials are highly encouraging.
5. Some of the salient features of the new approach, as a practical method, are enumerated.
6. This is the first paper on the new approach. The conceptual development and the on-computer implementation have progressed to a sufficient level of sophistication that the new approach may be explored for all its possibilities in a serious academic study.

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A Note about Intellectual Property Rights

The first author asserts his moral right to international patents based on the new approach.

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